Causal Impact Modeling of State Dependent Impulsive Affine Systems using Non-Standard Analysis

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Abstract—A causal modeling for an impulsive system with a state dependent switching surface is defined and analyzed on a new extended real space, denoted as Krylov hyperreals, which is based on the nonstandard analysis (NSA). The recent work of the authors contains the detailed construction of the extended space, and the generalized function on that space. In the first part of the paper, important concepts of NSA, and the suggested function space are reviewed. Next, a new generalization of a continuous but not differentiable function will be defined on the Krylov hyperreals in order to properly define a composition between singular and non differentiable function. By using an analogy to a spring and damper model, the authors suggest an equivalent causal model of the state dependent impulsive system by introducing the powers of singular control in the original continuous dynamics. A motivational example of a bouncing ball moving on a horizontal surface is analyzed to show the effectiveness.

I. INTRODUCTION

A model of mechanical systems which interacts with the environment by an impacting behavior involves an instantaneous change in their motion, usually in the velocity, at the time of the impact. Classically, such a system is modeled by an impulsive system, which contains a single continuous dynamical part, and a discrete dynamical part that models the resetting events of the state. The classical theory of impulsive systems may be referred to as an effect system since the discrete jump dynamics shows only the result of the impact not the cause of this effect. Fundamental theory on the impulsive systems can be found in [1].

In this paper, we are interested in utilizing singular controls in the original continuous dynamics to find the equivalent causal representation which generate the same jump dynamics as in the effect system. The objective is to better obey the physics in the model rather than approximate by the discrete jumps in the effect system. An application of controlling the impact can be found in [2], which analyzes the impact force in a relatively short duration to generate desired gait for a quadrupedal robot.

Nevertheless, applying a singular control, as defined by Schwartz distribution, encounters many difficulties for nonlinear systems since the multiplication between singular functions and a discontinuous test functions are not well defined, and powers of singular function cannot be well defined in the distribution: Schwartz impossibility theorem [3]. Therefore, most of the work on modeling the singular control in nonlinear system theory has been done by first regularizing the singular control system, and then sequentially approximating the solution to generalize the solution. Selected analysis of singular control in nonlinear systems can be found in [4]–[10] where the measure driven model is used in [4], [5]. A sequentially defined infinitesimal model was proposed in [6]–[9]. Although most of the paper provides a generalized solution of the impulse driven system, the limiting process itself lacks physical meaning. In this paper, we propose a new generalized solution which can avoid using the limiting process but rather define the solution point-wise at each infinitesimal moments of impact in Non-standard Analysis (NSA) framework, [11].

The key idea of NSA is to extend the real space so that the infinitesimals and infinity can be formally defined as a proper element in the extended space: hyperreal space. The infinitesimal elements and infinitely large elements are not unique, and there are uncountably many. Since NSA is less known to the control system society, up to our knowledge, we briefly recap important definitions in Section III-A. Detailed reference for NSA can be found in [12]. The usage of NSA in the generalized function theory can be found in [13] which translate a Colombeau algebra, a quotient algebra on a set of generalized function, [14], into the NSA framework. These works shows that the multiplication between nonsmooth and singular functions are well-defined either on hyperreals or in a Colombeau algebra. An application to the impulsive equations with an initial condition at the singular moment is considered in [15].

Recently in [16], we proposed a simple and intuitive way to extend the reals with an algebraically structured countably infinite dimensional subspace of original hyperreal space. In this space, we consider only countably many infinitesimals which are generated by geometric sequences with a different ratio. Therefore, by using the successive operation, similar to the Krylov method in numerical linear algebra but with the scaling and translation operators, we generalized the function space. Two important features of using the new generalized function is that the space is closed under multiplication, and the discontinuous function can be continuized in the infinitesimal time with a certain shape function. In this paper, we extend our previous results to analyze the impulsive system which the switching time depends on the state values. The first step is to extend the space of shape functions in order to properly define a composition between \( \delta \) function and a continuous but non differentiable function at a point of the impact. This composition is necessary when the switching surface is state dependent.

A motivational example of a bouncing ball moving on a
frictionless horizontal plane will be analyzed. A different contact force model for non-purely elastic collisions has been studied in [17] by having position dependent spring and damper coefficients (usually the powers of penetration position). In this paper, we also propose a nonlinear spring and damper model where a position dependent singular function generates the contact force which is capable of resetting to the desired velocity after the inelastic collision. Similarly, a regularization for inelastic bouncing ball in infinitesimal dynamics can be found in [7] but the singular control was not composed with the state.

In Section II, a motivating example of the bouncing ball is introduced, and in Section III, a brief review of NSA, and the framework of generalized function will be revisited. In Section IV, an extended notion of the shape space, and the general form of nonsmooth function will be proposed, and in Section V, a physically inspired regularization of bouncing ball problem will be analyzed, and the equivalent causal model of the classical impulsive system will be proposed with the solution of corresponding shape functions.

NOMENCLATURE

\( (r; s) \) a geometric sequence: \( \{s \cdot r^n\}_n \) for \( s, r \in \mathbb{R} \)

\( \{a_n\}_n \) a real valued sequence: \( a_n \in \mathbb{R} \) for all \( n \in \mathbb{N} \)

\( S_n \) a scaling operator: \( S_n(f)(t) = f(n t) \)

\( T_n \) a translation operator: \( T_n(f)(t) = f(t - \alpha) \)

II. MOTIVATIONAL EXAMPLE

Suppose that we have a bouncing ball moving on a one dimensional horizontal frictionless space. The state space is defined by \( (x(t), v(t))^T \), where \( x(t) \) is the position, and \( v(t) \) is the velocity of the ball. Fig 1 graphically shows the reference frame for \( x \), and we assume that the ball is a point mass, and so the ball hits the wall when \( x(t) = 0 \).

Let \( X = (x_1, x_2)^T \in \mathbb{R}^2 \) be the state space where \( x_1 := x \) and \( x_2 := v \). The impulsive effect model of this system with the restitution coefficient \( \gamma \in (0, 1) \), is given by

\[
\begin{align*}
\dot{X}(t) &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} X(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t) \quad \text{if } X(t) \not\in S \\
\Delta X(t) &= \begin{pmatrix} 0 \\ - (\gamma + 1) x_2(t^-) \end{pmatrix} \quad \text{if } X(t) \in S
\end{align*}
\]

(1)

where the switching surface is \( S := \{X \in \mathbb{R}^2 | x_1 = 0 \} \), and \( \Delta X(t) \) represents the difference, \( X(t^+) - X(t^-) \). The jump equation can also be written as \( x_2(t^-) = -\gamma x_2(t^-) \) where \( x_2(t^-) \) and \( x_2(t^-) \) are defined as the right limit and the left limit of \( x_2 \) at time \( t \). In addition, \( u(t) = 0 \) since the surface is frictionless.

Our objective is to find a generalized singular function, \( u^* \), such that Eqn (1) can be represented by

\[
\dot{X}(t) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} X + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u^*
\]

(2)

where \( u^* = u + k(\delta(x_1)) \) for some smooth function \( k : \mathbb{R} \rightarrow \mathbb{R} \). If there exist a smooth function \( k \), and if \( \delta(x_1) \) can be viewed as a composition of a distribution and a smooth function, then we can rewrite

\[
\delta(x_1) = \frac{\delta(t - t_0)}{|x_1(t_0)|},
\]

(3)

where \( t_0 \) represents the impact time, see [18]. However, the composition of \( \delta \circ x_1 \) is not well defined since \( x_1 \) is not differentiable at time \( t_0 \): \( \dot{x}_1(t_0^-) \neq \dot{x}_2(t_0^+) \).

Therefore, in this paper, we generalize a continuous but non-differentiable function to be differentiable in the infinitesimal hyperreal space by proposing a new space of shape functions addition to the one in [16]. This will give a way to properly define the \( \delta(x_1) \) in the above example.

III. NONSTANDARD SPACE TIME

In this section, a brief introduction to the hyperreals and important properties of NSA will be recapped. In addition, an algebraically structured infinite dimensional hyperreal space, and the generalized function on the extended space proposed in [16] will also be summarized.

A. Nonstandard analysis

In NSA, a real space is extended to a larger space by using a procedure similar to the construction of real space generated by the equivalence classes of the set of rational Cauchy sequences. A set \( \mathbb{R}^\mathbb{N} \) is defined as a set of real valued sequences. By properly defining the equivalence class, \( \sim \), using the filter in NSA, the quotient space, \( \mathbb{R}^\mathbb{N} \sim \), is defined as a hyperreal space, and let \( \langle r_n \rangle \) denote an equivalence class of \( \{r_n\}_n \in \mathbb{R}^\mathbb{N} \sim \). Essentially, different rates of convergence yield different equivalence classes.

Definition 1. (Hyperreal space)
A quotient space \( {}^*{\mathbb{R}} := \{ \langle r \rangle : \langle r_n \rangle \in \mathbb{R}^\mathbb{N} \} \) is called the extended real space or hyperreal space, and the members of \( {}^*{\mathbb{R}} \) are called hyperreal numbers.

The ordering property, addition and multiplication are well defined in \( {}^*{\mathbb{R}} \) element-wise, and it is proved in [12] that \( {}^*{\mathbb{R}} \) is an ordered field. See Theorem 3.6.1 in [12]. In addition, the infinitesimal can be formally defined as an element in \( {}^*{\mathbb{R}} \).

Definition 2. (Infinite, limited, and unlimited number)
An element \( \langle e_n \rangle \in {}^*{\mathbb{R}} \) is called infinitesimal if \( A = \{n \in \mathbb{N} : e_n < r \} \) is co-finite, i.e., \( \mathbb{N} \setminus A \) is finite, for any positive real \( r \in \mathbb{R}^+ \), and \( (0) < (|e_n|) \). An element \( b := \langle b_n \rangle \) is limited if there exist \( M > 0 \) such that \( (|b_n|) < (M) \). An element \( \langle E_n \rangle \in {}^*{\mathbb{R}} \) is called unlimited if \( (|E_n|) > (r) \) for any \( r \in \mathbb{R} \).

Lastly, given a limited hyperreal number there exist real number \( c \in \mathbb{R} \) such that \( b - c \) is an infinitesimal.
Definition 3. (halo and shadow)
Given a hyperreal number $b \in ^*\mathbb{R}$, a subset $\text{hal}(b) \subset ^*\mathbb{R}$ is called halo of $b$ if $\text{hal}(b) = \{ c \in ^*\mathbb{R} : b - c$ is infinitesimal$\}$, and $c \in \mathbb{R}$ is called a shadow of $b$ if $b - c$ is infinitesimal, and we define it as $sh(b)$.

The set of infinitesimals can be written as $\text{hal}(0)$, and any zero converging sequence is an infinitesimal. For example, a geometric sequence $\langle 1/2; 1 \rangle$ is an example of infinitesimal.

B. Krylov hyperreal space
In this section, a theory of an algebraically structured countably infinite dimensional subspace of $^*\mathbb{R}$ is recapped. Throughout this paper, assume that $\alpha > 1$.

Definition 4. ($i$-th Krylov basis and $^*\mathbb{R}^N$ space)
Given $i \in \mathbb{Z}$. A sequence $e_i = \langle (\frac{1}{i})^j ; 1 \rangle$ is called a $i$-th Krylov basis. Furthermore, a set $^*\mathbb{R}^N = \{ x \in ^*\mathbb{R} | x = \sum_{i=0}^{N} s_i \cdot e_i \}$ where $\{ s_i \}_{i=0}^{N} \subset \mathbb{R}$ is called $^*\mathbb{R}^N$ space where $N \in \mathbb{N}$.

The Krylov base vectors, $\{ e_i \}_{i}$, are linearly independent in $^*\mathbb{R}^N$ (Proposition 11 in [16]), and by letting $N$ go to infinity, a generalized countably infinite dimensional hyperreal space can be defined.

Definition 5. ($^*\mathbb{R}$ space)
A set $^*\mathbb{R} = \{ x \in ^*\mathbb{R} | x = \sum_{i=0}^{\infty} s_i \cdot e_i \}$ where $\{ s_i \}_{i=0}^{\infty} \subset \mathbb{R}$ is called a Krylov hyperreal space where $l_1$ is a space of absolutely convergent real valued sequence.

Observe that given a hyperreal element $t^*$ in $^*\mathbb{R}$, one can represent $t^*$ with a sequence $\{ s_i \}_{i}$, and the shadow point of $t^*$ corresponds to the first element, $sh(t^*) = s_0$. In addition, all the other elements of the sequence, $s_i e_i$, are infinitesimal, so we can decompose $t^*$ to the shadow part, $s_0$, and the remainder parts depending on the convergence speed.

Definition 6. (Insensible and sensible time)
Given a $t^* \in ^*\mathbb{R}$ with a representation, $\{ s_i \}_{i} \in l_1$. The component $s_0$ is called sensible time of $t^*$, and $s_i$ is called insensible time of $t^*$ with $1/\alpha^i$ as a rate of convergence.

C. Krylov function space
A generalized function is now defined on the Krylov hyperreal space, $^*\mathbb{R}$.

Definition 7. (Hyperreal function and Krylov function)
A mapping $F : ^*\mathbb{R} \rightarrow ^*\mathbb{R}$ is called a hyperreal valued function. If there exist a sequence of functions, $\{ f_n \}_{n} \subset \mathbb{R}^\mathbb{R}$, such that $\sigma_{t}, F \equiv \langle f_n(t_n) \rangle$ where $t^* \in ^*\mathbb{R}$ and $t^* \equiv \langle t_n \rangle$. We denote the hyperreal valued function as $F := \langle f_n \rangle$. In addition, the restriction of $F$ to the Krylov space, $^*\mathbb{R}$, is called a Krylov function.

Similar to finding a shadow point in $^*\mathbb{R}$, we can also find a regular function $f \in \mathbb{R}^\mathbb{R}$ which is defined by $f(t) := sh(\sigma_{t}, \langle f_n \rangle)$ for each $t = sh(t^*)$. The corresponding regular function is called an $\mathbb{R}$-sampled function.

Definition 8. ($^*\mathbb{R}$-sampling)
Given a Krylov function $F : ^*\mathbb{R} \rightarrow ^*\mathbb{R}$ generated by a pointwise convergent sequence of continuous functions, a mapping $f : \mathbb{R} \rightarrow \mathbb{R}$ is called an $\mathbb{R}$-sampled function of $F$ if, for any given $s_0 \in \mathbb{R}$, the evaluation $f(s_0)$ is equal to $sh(\sigma_{t}, F)$ for some $t^* \in ^*\mathbb{R} \cap \text{hal}(s_0)$.

Let $\chi$ be a characteristic function, and suppose that a Krylov function is given as $F = \langle S_{\alpha} u \rangle$ for some $\alpha > 1$, and $u(t) = f_{\chi[0,1]}(t) + \chi(1,\infty)(t)$ which is a ramp function on $[0, 1]$, then the $\mathbb{R}$-sampled function of $F$,

$$f(s_0) = sh(\sigma_{t}, F) = \begin{cases} 0 & s_0 < 0 \\ s_0 = 0 & 1 \; s_0 > 0 \end{cases}$$

is a Heaviside function with a value $u(s_1)$ at $s_0 = 0$ for some $s_1 \in \mathbb{R}$. Important properties of having a generalized Heaviside function $F$ is that (1) every $\mathbb{R}$-sampled function corresponds to the regular Heaviside function on $\mathbb{R}$, (2) there are infinitely many different generalized Heaviside function $F$ depends on the shape of $u$ and the speed $\alpha$, and (3) the generalized Heaviside function is continuous on the extended insensible time while it is still discontinuous on the sensible time after taking $\mathbb{R}$-sampling process. Here we recap the extended scaling and translation operator, and give a full definition of generalized Heaviside function. Assume that $F^*$ is a set of Krylov functions.

Definition 9. ($S_{<\alpha^1;1}$ and $T_{<\alpha^1;1}$ operator)
A mapping $S_{<\alpha^1;> : F^* \rightarrow F^*$, where $\langle a_n \rangle \in ^*\mathbb{R}$, is called a Krylov scaling operator if, given $F \in F^*$ where $F = \langle f_n \rangle$, the evaluation of $S_{<\alpha^1;> (F)$ is $\langle S_{\alpha} f_n \rangle$. Similarly, a mapping $T_{<\alpha^1;> : F^* \rightarrow F^*$, where $\langle a_n \rangle \in ^*\mathbb{R}$, is called a Krylov shifting operator if, given $F \in F^*$ where $F = \langle f_n \rangle$, the evaluation of $T_{<\alpha^1;> (F)$ is $\langle T_{\alpha} f_n \rangle$.

Definition 10. (Shape function)
A function $u \in C(\mathbb{R})$ is called a shape function if there exist a function $p \in C^1(\mathbb{R})$ where $p$ is strictly monotonic in $[0, 1]$ such that $p(0) = 0$ and $p(1) = 1$, so that $u = \chi(0,1)^{\mathcal{D}} + \chi(1,\infty)$.

Definition 11. (Krylov Heaviside function)
A function $F \in F^*$ is called a Krylov Heaviside function if there exist a shape function $u$ and $N \in \mathbb{Z}$ such that $F = S_{<\alpha^N;1} u$. In addition, $\alpha^N$ is called a rate of $F$.

A generalized singular delta function is derived in [16] by differentiating the Krylov Heaviside function $F$ for each element function. Since the derivative of each element function is well defined in the insensible time, it is possible to define the delta function in infinite dimensional Krylov space, $^*\mathbb{R}$.

Definition 12. (Krylov Delta function)
A function $\delta \in F^*$ is called a Krylov delta function, and defined as

$$\delta = \langle \alpha;1 \rangle (T_{\alpha} f_{\chi[0,1]}^*) = S_{<\alpha^2;1} u,$$

which is a derivative of some Krylov Heaviside function, $F = S_{<\alpha^N;1} u$. 


In addition, the powers of Krylov delta function, $\delta^n$, and its higher derivative, $\delta^{(n)}$ are also point-wise defined in $\mathbb{R}$ for all $n \in \mathbb{N}$.

$$
\delta^n = (\alpha^n; 1)(I - T_{1/2})S_{(\alpha^2; 1)}u^n \quad (6)
$$

$$
\delta^{(n)} = (\alpha^{2n+1}; 1) \prod_{i=0}^{n}(I - T_{1/(\alpha^{2i+1}; 1)})S_{(\alpha^{2n+1}; 1)}u \quad (7)
$$

**IV. GENERALIZED NONSMOOTH FUNCTION**

Suppose that $x$ is a real valued piecewise continuous function, then $x$ can be decomposed into two parts,

$$
x = x_c + \sum_{i=1}^{N} a_i T_{\tau_i}S_{(\alpha^{n(i)}; 1)}u_i, \quad (8)
$$

where $x_c$ is a continuous function, $\{\tau_i\}_{i=1}^{N}$ are points of discontinuity, and $\{a_i\}_{i=1}^{N}$ are the amount of jumps where $N$ is the total number of discontinuity. The function, $u_i$, and $n(i)$ represents the corresponding shape function, and the convergence rate, respectively, for each $i$. Observe that the Krylov Heaviside part in Eqn (8) contains the shape function which shows how the discontinuous points will be connected in insensible time. In addition, the derivative of the Heaviside part can be obtained from the shape definition but the continuous part do not have the information on how the connection will be made for $x_c$ in sensible time. If $x_c$ has a point that is not differentiable, then the Eqn (8) is insufficient to generalize the derivative of the $x$ in sensible time. Therefore, we propose a new shape function which can generalize a continuous but nonsmooth function.

**A. Self returning shape function (bump)**

The main objective of introducing the shape function in Definition 10 is to find a continuous connection between two discontinuous points, and make the limit of the Krylov scaling operator act on the shape function convergence pointwise to a Heaviside function in sensible time. For example, let $f_n = S_{\alpha^n}u$ where $u(t) = \chi_{(0,1)}^c + \chi_{(1,\infty)}^c$ is the ramp shape function, then we have $\lim_{n \to \infty} f_n(t) = 1$ for all $t > 0$, and $\lim_{n \to \infty} f_n(t) = 0$ for all $t < 0$. This convergence result guarantees that the discontinuity model in Eqn (8) will generate the desired jump for $x_c$ in sensible time.

Similar to the construction of the original shape function, we introduce a new shape function which connects from $u(0) = 0$ to $u(1) = 0$. Our goal is to find a smooth connection between the left derivative and right derivative of a non differentiable function, $x_c$, in sensible time. For example, $u(t) = \chi_{(0,1)}^c \sin(\pi t)$ can be a new shape function since $u(0+) = \sin(0) = 0$ and $u(1-) = \sin(\pi) = 0$. By the mean value theorem, there exist at least one point $t_1$ in $(0, 1)$ such that $\dot{u} = 0$. Therefore, except for the trivial solution where $u \equiv 0$, there must be a bump within $(0, 1)$. A new shape function is now introduced by using this necessary conditions on returning to zero.

**Definition 13. (Bump shape function)**

A function $u \in C^1(\mathbb{R})$ is called a bump shape function if there exist a function $p \in C^1(\mathbb{R})$ such that $p(0) = 0$ and $p(1) = 0$ and $u = \chi_{(0,1)}^c p$.

**B. Generalized nonsmooth function**

Suppose that $x_c$ in Eqn (8) is continuous but not differentiable at $t = 0$, and the left limit $\dot{x}_c(0^-)$, and the right limit $\dot{x}_c(0^+)$ of the derivatives exist but not equal. The objective of this section is to properly define the generalized form for $x_c$ using the new shape function in Definition 13, which will end up having a smooth connection between the discontinuity of derivatives.

By directly applying the Krylov scaling operator to the new bump shape function $u$, we can postulate a new generalized function $\pi_c := x_c(t) + S_{(\alpha^1)1}u$ as an analogy of constructing Krylov Heaviside function. Let the derivative of $\pi_c$ at $t = 0$ to be defined as $D\pi_c = D(S_{(\alpha^1)1})u$ where $D$ is the derivative operator. By evaluating the derivative for $\pi_c$ at $t^* = 0c + s_1e_1$ for some $s_1 \in (0, 1)$ we have $D\pi_c := (\alpha^1)D u(s_1)$. However, this shows that $D\pi_c$ goes to infinity, and so $x_c$ is not continuous, which contradicts the fact that $x_c$ is continuous. A remedy to this problem is by defining the generalized $\pi_c$ in the next form.

$$
\pi_c := x_c(t) + (1/\alpha^1) S_{(\alpha^1)1}u. \quad (9)
$$

For example, suppose that $x_c(t)$ is given as in the top graph in Fig 2 that $x_c$ is differentiable everywhere except at $t = 0$. By evaluating the above function at Krylov time, $t^* = s_0e_0 + s_1e_1$, we can see that

$$
\sigma_t \cdot \pi_c := \sigma_t \cdot (x_c) + \sigma_t \cdot (1/\alpha^1) S_{(\alpha^1)1}u
$$

$$
= \sigma_t \cdot x_c + \epsilon
$$

where $\epsilon \in \text{hal}(0)$ is an infinitesimal. The equality holds since $u$ is continuous and it is bounded on $(0, 1)$, which then the multiplication with $(1/\alpha^1)$ makes the right term be an infinitesimal. Therefore, the $\mathbb{R}$-sampling of $\pi_c$ is equivalent to $x_c$ in the real time. The only requirement for the bump shape function $u$ is to satisfy the boundary conditions:
\(Du(0)\) and \(Du(1)\) should be equal to \(\dot{x}_c(0^-)\) and \(\dot{x}_c(0^+)\), respectively.

Now define a generalized derivative of Eqn (9) at the non-differentiable point \(sh(t^*) = 0\), and evaluate at \(t^* = 0_{e0} + s_1 e_1\) for \(s_1 \in (0, 1)\), then

\[
\sigma_{t^*}D\pi_c := \sigma_{t^*}(1/\alpha; 1) \langle \alpha; 1 \rangle S_{(\alpha; 1)}Du
\]

\[
\cong \sigma_{t^*}(S_{(\alpha; 1)}Du)
\]

where the last equality is in the sense of equivalent class. By using this generalized derivative, we now define a generalized nonsmooth function as

\[
x = x_c + \sum_{i=1}^M T_k\langle 1/\alpha; 1 \rangle S_{(\alpha; 1)}w_i + \sum_{i=1}^N \alpha_i T_r S_{(\alpha_n; 1; 1)}u_i,
\]

where \(M\) is the number of non-differentiable point of \(x_c\), and \(\{w_i\}_{i=1}^M\) are corresponding bump shape functions. Since the smooth change in the derivative is now well defined, we are able to evaluate point-wise the composition between \(\delta\) function in Eqn (5), and a non smooth function \(x\) in Eqn (9) so that we can use to solve the generalized ordinary equation of the form in Eqn (2).

V. BOUNCING BALL EXAMPLE

A. Physically inspired model

The objective is to represent the effect model Eqn (1) with corresponding causal model in Eqn (2). We propose that the goal can be achieved by modeling the generalized force, \(u^*\), with a nonlinear spring and damper model having position dependent coefficients, \(k(x_1)\) and \(c(x_1)\), respectively. We model that these coefficients are unlimited number when \(x_1 = 0\), which represents the hard wall model in sensible time. On the other hand, these coefficients can also be understood as a sequence of nonlinear state dependent spring and damper in insensible time. Therefore, The suggested generalized force is modeled as

\[
u^* = -k(x_1)x_1 \quad + c(x_1)x_2
\]

\[
k(x_1) = \frac{\pi^2}{1 - \zeta^2} \delta^2(x_1)
\]

\[
c(x_1) = \frac{2\pi\zeta}{1 - \zeta^2} \delta(x_1),
\]

where \(\zeta\) corresponds to the damping ratio of the insensible spring and damper model. By choosing \(\zeta = -\ln(\gamma)/\pi\), we have \(\zeta \in (0, 1)\) for \(\gamma \in (e^{-\pi}, 1)\) which means that the system is an under-damped system. In the next section, we show that the Krylov based singular control, Eqn (11), generates the same jump equation in Eqn (1). In the proposed model, we choose \(\delta\) defined in Eqn (5), and the \(\delta^2\) defined in Eqn (6) with the ramp shape function \(u\).

B. Krylov based solution

In this section, we solve the Eqn (2) with an initial condition \(X(t_0) = (x_0, v_0)^T \in \mathbb{R}^2\) where \(x_0 > 0\) and \(v_0 < 0\). This shows that the ball initially moves towards the wall with constant velocity, and so \(x_1(t) = v_0(t - t_0) + x_0\). The initial time for the collision is at \(t_1 = t_0 - x_0/v_0\). For simplicity, assume that \(t_0 = x_0/v_0 < 0\) so that \(t_1 = 0\), and assume that \(\gamma \in (e^{-\pi}, 1) = (0.043, 1)\) to make the system under-damped. Now by using the general form of the nonsmooth formation in Eqn (10), we propose the solution as

\[
x_1 = x_{c1} + (1/\alpha; 1) S_{(\alpha; 1)}w_1
\]

\[
x_2 = x_{c2} + a_1 S_{(\alpha; 1)}w_2
\]

where \(w_1\) is some bump shape function, \(w_2\) is the original shape function, \(x_{c1}\) and \(x_{c2}\) are continuous function, and \(a_1\) is the corresponding instantaneous jump of \(x_2\) in the system. Let \(X_c = (x_{c1}, x_{c2})\). By taking the generalized derivative, we have

\[
\dot{X} = \dot{X}_c + \left( \frac{S_{(\alpha; 1)}Du_1}{\alpha (1/\alpha; 1) S_{(\alpha; 1)}Du_2} \right).
\]

The right hand side of Eqn (2) can also be computed as

\[
x x_{c2} + a_1 S_{(\alpha; 1)}w_2
\]

\[
-x(k(x_1)x_1 - c(x_1)x_2)
\]

Now we evaluate the Eqn (16) and Eqn (17) at the Krylov time \(t^* = s_0 e_0 + s_1 e_1\), and solve for \(a_1, w_1, w_2, c_{x1}\), and \(x_{c2}\) by using the boundary conditions for the shape functions, \(w_1(0) = w_1(1) = w_2(0) = 0\) and \(w_2(1) = 1\), and the initial condition, \(x_{c1}(0^-) = 0\) and \(x_{c2}(0^-) = 0\). For the case in the sensible time \(t^* = 0_{e0} + s_1 e_1\) for \(s_1 \in (0, 1)\), the generalize derivative for \(x_1\) gives that

\[
S_{(\alpha; 1)}Du_1 = v_0 + a_1 S_{(\alpha; 1)}w_2
\]

so by taking the left inverse operator, \(S_{(1/\alpha; 1)}\), we have

\[
Du_1 = a_1 w_1 + v_0.
\]

Similarly, by multiplying \((1/\alpha; 1)\cdot(1/\alpha; 1) S_{(1/\alpha; 1)}\) to the left and right for \(x_{c2}\), we have,

\[
Du_2 = \frac{-(1/\alpha; 1) S_{(\alpha; 1)}k(x_1)}{a_1} \cdot (S_{(1/\alpha; 1)}x_{c1} + (1/\alpha; 1) w_1)
\]

\[
-\frac{(1/\alpha; 1) S_{(\alpha; 1)}c(x_1)}{a_1} \cdot (S_{(1/\alpha; 1)}x_{c2} + a_1 w_2)
\]

Since we desire \(a_1\) to be equal to \((\gamma + 1) v_0\) in Eqn (1), so assume that \(v_0 + a_1 > 0\) holds. We can further simplify the above equation by evaluating \((1/\alpha; 1) S_{(1/\alpha; 1)}\delta(x_1)\) at \(t^* = 0_{e0} + s_1 e_1\) as

\[
\sigma_{s_1}((1/\alpha; 1) S_{(\alpha; 1)}((\alpha; 1) (I - T_{(1/\alpha; 1)}) S_{(\alpha; 1)}u \circ x_1)
\]

\[
= \sigma_{s_1}(I - T_{(1/\alpha; 1)}) S_{(\alpha; 1)}u \circ x_1 = 1
\]
for all $s_1 \in (0, 1)$ since $x_1(s_1) = (v_0 + a_1)s_1 > 0$. Similarly, 
\[ \langle 1/\alpha; 1 \rangle S_{(1/\alpha; 1)} \delta^2(x_1) \] is evaluated at $t^* = 0 = s_1 e_1$ as
\[
\begin{align*}
\sigma_{s_1} & \langle 1/\alpha; 1 \rangle S_{(1/\alpha; 1)} \langle (\alpha^2; 1) (I - T(\frac{\pi}{\alpha}; 1)) S_{(\alpha^2; 1)} u^2 \circ x_1 
= \sigma_{s_1} \langle 1/\alpha; 1 \rangle (I - T(1; 1)) S_{(1/\alpha; 1)} u^2 \circ x_1
\end{align*}
\]

Therefore, $Dw_2$ is now simplified to
\[
Dw_2 = -\frac{\pi^2}{a_1 (1 - \zeta^2)} w_1 - \frac{2\pi\zeta}{a_1 (1 - \zeta^2)} (v_0 + a_1 w_2)
\tag{19}
\]

By augmenting Eqn (18), we have the ordinary differential equation for the shape functions,
\[
\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 0 \\ -\zeta \end{pmatrix} - \frac{a_1}{(1 - \zeta^2)} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} + \begin{pmatrix} -\frac{2\pi\zeta}{a_1 (1 - \zeta^2)} v_0 \\ \frac{2\pi\zeta}{a_1 (1 - \zeta^2)} v_0 \end{pmatrix}.
\]

The two point boundary condition for $w_1$ and $w_2$ gives a unique solution,
\[
\begin{align*}
w_1(s_1) &= \frac{v_0}{\pi} e^{-\pi\zeta s_1} \sin(\pi s_1) \\
w_2(s_1) &= -v_0 (1 + e^{-\pi\zeta s_1} (\zeta \sin(\pi s_1) - \cos(\pi s_1))) / a_1
\end{align*}
\]

As expected, the bump shape function $w_1$ matches the zero boundary conditions, $w_1(0) = w_2(0) = 0$. By matching the boundary condition for the original shape function, $w_2$, we have $w_2(0) = 0$ and $w_2(1) = -v_0 (1 + e^{-\pi\zeta}) / a_1 = 1$. Therefore, we have
\[
a_1 = v_0 (1 + e^{-\pi\zeta}) = -v_0 (1 + e^{\ln(\gamma)}) = -v_0 (1 + \gamma)
\]

since $\zeta = -\ln(\gamma) / \pi$ was given initially. By substituting $a_1$ to generate the solution for $w_2$, we have
\[
w_2(s_1) = \frac{1 + e^{-\pi\zeta s_1} (\zeta \sin(\pi s_1) - \cos(\pi s_1))}{1 + \gamma}
\]

This shows that the shape function $w_2$ smoothly connects from 0 to 1 in insensible time $s_1 \in (0, 1)$, and the effect of singular control for the causal representation in Eqn (2) is $\Delta x_2(t) = -(1 + \gamma)v_0$, which is the desired jump in Eqn (1).

The graph of the bump shape function, $w_1$, is shown in Fig 3.

In addition, the necessary condition for the correct bump shape function holds since $\dot{w}_1(0) = v_0 = \dot{x}_1(0)$, and $\dot{w}_1(1) = a_1 + \gamma a_1 = \dot{x}_1(1)$. The negative value of $w_1$ shape function shows that in insensible time the ball moves towards to the negative value, and pulls it back to the original position at $s_1 = 1$.

VI. CONCLUSIONS

In this paper, a causal model for impulsive affine systems has been extended to cases where the impulsive behavior occurs on a submanifold of the state space. The framework so-called Krylov space and Krylov generalized function has been revisited, and extended to incorporate the representation of the nonsmooth function by introducing a bump shape function. The bouncing ball problem in one dimension (no gravity) has been analyzed in this new framework. The result shows the feasibility of utilizing the composition of a singular function and a non-smooth function, and is used in finding the causal model of the impulsive affine system.

REFERENCES


